Logarithmic p-Convex Functions and Some of Their Properties

Gültekin Tınaztepe[*], Sevda Sezer[†], Zeynep Eken[‡], Gabil Adilov[§]

October 13, 2021

Abstract

In this paper, the concept of logarithmic p-convex function is introduced. Then, fundamental characterizations and some operational properties of logarithmic p-convex functions are presented.

Keywords: Convex Function, p-Convex Function, Logarithmic p-Convex Function

AMS Subject Classification: 26A51, 26B25

1 Introduction

Convex functions are of special interest due to their nice properties regarding optimization problems, which can be defined on n-dimensional Euclidean space as follows:

Let $f : \mathbb{R}^n \to \mathbb{R}$. f is said to be convex function if

$$f(\lambda x + \mu y) \leq \lambda f(x) + \mu f(y) \tag{1}$$

for all $x, y \in \mathbb{R}$ and $\lambda, \mu \in [0, 1]$ such that $\lambda + \mu = 1$.

Since its emergenge on stage, the reseachers' enthuiasm and the requirements of novel problems in science have yielded to different kinds of convexity such as quasi convexity, exponential convexity, B-convexity, B^{-1}-convexity, r-convexity, s-convexity, p-convexity [1–12]. The logarithmic convexity is one of the most prominent types, which is defined as the convexity of the logarithm of a function, i.e., for a function $f : \mathbb{R} \to \mathbb{R}$, f is called logarithmically convex if $\log f$ is convex. As far as we reviewed the literature, the first appearance of this concept goes back to studies on gamma function of Artin [13], who first used the term

[*]Vocational School of Technical Sciences Akdeniz University Antalya-Turkey e-mail:gtinaztepe@akdeniz.edu.tr

[†]Faculty of Education Akdeniz University Antalya-Turkey e-mail:sevdasezer@akdeniz.edu.tr

[‡]Corresponding author, Faculty of Education Akdeniz University Antalya-Turkey e-mail:zeynepeken@akdeniz.edu.tr

[§]Faculty of Education Akdeniz University Antalya-Turkey e-mail:gabil@akdeniz.edu.tr

logarithmic convexity. The basic characterizations and properties of the *log*-convex functions can be found in [14–18]. Among its application areas can be counted geometric programming in optimization, structural stability issues in thermoelasticity theory, growth theory and modelling of some inference-coupled multiuser sytems in information theory [14, 19–22].

In this paper, we introduce a novel logarithmic convexity associated with p-convexity. Briefly, p-convexity of a function is defined on a p-convex set and obtained by putting certain conditions on parameters λ, μ in (1). In literature, the definition of p-convex set has been introduced quite earlier than p-convex functions [23]. For the sake of clarification, let us recall them:

Let $p \in [0, 1]$ and A be subset of the vector space X. A is said to be p-convex if

$$\lambda x + \mu y \in A \tag{2}$$

for all $x, y \in A$ and $\lambda, \mu \in [0, 1]$ such that $\lambda^p + \mu^p = 1$.

p-convex set is a generalization of convex set with some differences. From the definition, it is trivial that a singleton is not a p-convex set but a convex set. Furthermore, an interval of real numbers is convex set, only an interval including zero or accepting it as boundary point can be p-convex set. In n-dimensonal Euclidean space $\mathbb{R}^n$, for a fixed point, a ray connecting origin to a point represents a p-convex set.

p-Convex function is introduced in [12], which is defined as follows:

Let A be p-convex set and $f : A \to \mathbb{R}$. f is said to be p-convex function if

$$f(\lambda x + \mu y) \leq \lambda f(x) + \mu f(y) \tag{3}$$

for all $x, y \in \mathbb{R}$ and $\lambda, \mu \in [0, 1]$ such that $\lambda^p + \mu^p = 1$. To illustrate, the function $f : \mathbb{R}_+ \to \mathbb{R}$ defined by $f(x) = x^2$ is a p-convex function [24]. Moregenerally, using Theorem 3.15 in [12], we obtain that the function f defined by $f(x) = x^{2^n}$ on $\mathbb{R}_+$ is a p-convex function for $n \in \mathbb{N}^+$.

It is clear that in case $p = 1$, p-convexity coincides with convexity. In [12], it is stated that in case $p \to 0$, p-convex set can be accepted as star convex set with respect to zero and p-convex function is considered as subhomogeneous function. The some of basic properties and characterizations of p-convex set and functions can seen also in [12, 23, 25, 26] and the references therein.

Various studies have been done on p-convex functions involving inequalities [24, 27, 28]. Furthermore, it has been defined different functions such as quasi p-convex and p-concave functions [29]. Definition of quasi p-convex and p-concave functions are given below:

Let $U \subseteq \mathbb{R}^n$ be a p-convex set. A function $f : U \to \mathbb{R}$ is called quasi p-convex function if f provides

$$f(\lambda x + \mu y) \leq \max\{f(x), f(y)\}$$

for each $x, y \in U$; $\lambda, \mu \geq 0$ such that $\lambda^p + \mu^p = 1$. If the function -f is quasi p-convex, f is called quasi p-concave function.

In this study, we introduce logarithmic p-convex (p-concave) functions and its basic characterizations. Also, preservation of logarithmic p-convexity on some algebraic oprations are examined. Interrelations among logarithmic p-concave and logarithmic p-convex and quasi p-convex functions are exposed.

2 Main Results

Throughout the paper, unless otherwise stated, $U \subseteq \mathbb{R}^n$ is a p-convex set, $\mathbb{R}_+ := [0, +\infty)$ and $\mathbb{R}_{++} := (0, +\infty)$.

Definition 2.1. *Let $f : U \to \mathbb{R}_{++}$. The function f is called logarithmic p-convex (p-concave) function if the function $logf$ is p-convex (p-concave). The logarithmic p-convex functions are denoted by log-p-convex (log-p-concave) functions for short.*

The following theorem gives us a characterization of *log*-p-convex functions:

Theorem 2.2. *The function $f : U \to \mathbb{R}_{++}$ is a log-p-convex function if and only if*

$$f(\lambda x + \mu y) \leq [f(x)]^{\lambda}[f(y)]^{\mu}$$

is satisfied for all $x, y \in U$, $\lambda, \mu \geq 0$ such that $\lambda^p + \mu^p = 1$.

Proof. ($\Leftarrow$:) It is clear from Definition 2.1.

($\Rightarrow$:) Let f be a *log*-p-convex function, then $\log f$ is p-convex. Thus we can obtain

$$(logf)(\lambda x + \mu y) \leq \lambda(logf)(x) + \mu(logf)(y) = log([f(x)]^{\lambda}[f(y)]^{\mu})$$

for all $x, y \in U$ and $\lambda, \mu \geq 0$ such that $\lambda^p + \mu^p = 1$. This shows that $f(\lambda x + \mu y) \leq [f(x)]^{\lambda}[f(y)]^{\mu}$, i.e., f is *log*-p-convex function. $\square$

Theorem 2.3. *Let $f : U \to \mathbb{R}_{++}$. The function f is a log-p-concave function if and only if*

$$f(\lambda x + \mu y) \geq [f(x)]^{\lambda}[f(y)]^{\mu}$$

is satisfied for all $x, y \in U$, $\lambda, \mu \geq 0$ such that $\lambda^p + \mu^p = 1$.

Proof. The proof is similar to the proof of Theorem 2.2. $\square$

Example 2.4. *Let $a \in (1, \infty)$ and $b \in \mathbb{R}$. The function $f : \mathbb{R}_+ \to \mathbb{R}_{++}$ defined by $f(x) = a^{bx}$ is a log-p-convex and log-p-concave function.*

One of the main properties of the convex function is that they satisfy the Jensen inequality. The following theorem shows that *log*-p-convex functions also satisfy the Jensen inequality.

Theorem 2.5. *Let $f : U \to \mathbb{R}_{++}$ be a log-p-convex function. Let $x_1, x_2, \ldots, x_m \in U$ and $\lambda_1, \lambda_2, \ldots, \lambda_m \geq 0$ with $\lambda_1^p + \lambda_2^p + \cdots + \lambda_m^p = 1$. Then*

$$f(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_m x_m) \leq [f(x_1)]^{\lambda_1}[f(x_2)]^{\lambda_2} \cdots [f(x_m)]^{\lambda_m}.$$

Proof. We use induction on m. The inequality is trivially true when $m = 2$. Assume that it is true when $m = k$, where $k > 2$. Now we show the validity when $m = k + 1$. Let a real number x be defined by the equation

$$x = \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_{k+1} x_{k+1}$$

where $x_1, x_2, \ldots, x_{k+1} \in U$ and $\lambda_1, \lambda_2, \ldots, \lambda_{k+1} \geq 0$ with $\lambda_1^p + \lambda_2^p + \cdots + \lambda_{k+1}^p = 1$. At least one of $\lambda_1, \lambda_2, \ldots, \lambda_{k+1}$ must be less than 1. Let us say $\lambda_{k+1} < 1$ and write

$$\lambda_1^p + \lambda_2^p + \cdots + \lambda_k^p = 1 - \lambda_{k+1}^p.$$

One can find $\lambda_* < 1$ such that $\lambda_1^p + \lambda_2^p + \cdots + \lambda_k^p = \lambda_*^p$. Since $(\frac{\lambda_1}{\lambda_*})^p + (\frac{\lambda_2}{\lambda_*})^p + \cdots + (\frac{\lambda_k}{\lambda_*})^p = 1$ and the assumption of hypothesis, we get

$$f\left(\frac{\lambda_1}{\lambda_*} x_1 + \frac{\lambda_2}{\lambda_*} x_2 + \cdots + \frac{\lambda_k}{\lambda_*} x_k\right) \leq [f(x_1)]^{\frac{\lambda_1}{\lambda_*}} [f(x_2)]^{\frac{\lambda_2}{\lambda_*}} \cdots [f(x_k)]^{\frac{\lambda_k}{\lambda_*}}.$$

By using *log*-p-convexity of f,

$$\begin{aligned} f(x) &= f(\lambda_*(\tfrac{\lambda_1}{\lambda_*} x_1 + \tfrac{\lambda_2}{\lambda_*} x_2 + \cdots + \tfrac{\lambda_k}{\lambda_*} x_k) + \lambda_{k+1} x_{k+1}) \\ &\leq [f(\tfrac{\lambda_1}{\lambda_*} x_1 + \tfrac{\lambda_2}{\lambda_*} x_2 + \cdots + \tfrac{\lambda_k}{\lambda_*} x_k)]^{\lambda_*} \cdot [f(x_{k+1})]^{\lambda_{k+1}} \\ &\leq [f(x_1)]^{\lambda_1} [f(x_2)]^{\lambda_2} \cdots [f(x_{k+1})]^{\lambda_{k+1}} \end{aligned}$$

is obtained. This completes the proof by induction. $\square$

Theorem 2.6. *Let $f : U \to \mathbb{R}_{++}$. For any $x, y \in U$, the function $\varphi : [0,1] \to \mathbb{R}_{++}$ defined by $\varphi(\lambda) = f(\lambda x + (1 - \lambda^p)^{\frac{1}{p}} y)$ is a log-p-convex function, then f is also a log-p-convex function.*

Proof. Let $x, y \in U$ and $\lambda \in [0,1]$. Then

$$\begin{aligned} f(\lambda x + (1 - \lambda^p)^{\frac{1}{p}} y) = \varphi(\lambda) &= \varphi(\lambda \cdot 1 + (1 - \lambda^p)^{\frac{1}{p}} \cdot 0) \\ &\leq [\varphi(1)]^{\lambda} [\varphi(0)]^{(1-\lambda^p)^{\frac{1}{p}}} \\ &= [f(x)]^{\lambda} [f(y)]^{(1-\lambda^p)^{\frac{1}{p}}}. \end{aligned}$$

$\square$

Theorem 2.7. *(i) If the function $f : U \to [1, \infty)$ is a log-p-convex function, then f is a quasi p-convex function.*

(ii) If the function $f : U \to (0, 1)$ is a log-p-concave function, then f is a quasi p-concave function.

Proof. (i) Let $x, y \in U$ and $\lambda, \mu \geq 0$ such that $\lambda^p + \mu^p = 1$. Assume that $\max\{f(x), f(y)\} = f(x)$, then we have

$$f(\lambda x + \mu y) \leq [f(x)]^{\lambda} [f(y)]^{\mu} \leq [f(x)]^{\lambda} [f(x)]^{\mu} = [f(x)]^{\lambda+\mu} \leq f(x) = max\{f(x), f(y)\}.$$

(ii) Quasi p-concavity of *log*-p-concave functions can be proved in the same way.

$\square$

Theorem 2.8. *If the functions $f_i : U \to \mathbb{R}_{++}$ are log-p-convex (log-p-concave) functions for $i = 1, 2, \cdots, m$, then $f = \prod_{i=1}^{m} f_i$ is a log-p-convex (log-p-concave) function.*

Proof. For $x, y \in U$ and $\lambda, \mu \geq 0$ such that $\lambda^p + \mu^p = 1$, we have

$$\begin{aligned} f(\lambda x + \mu y) &= \prod_{i=1}^{m} f_i(\lambda x + \mu y) \\ &\leq \prod_{i=1}^{m} \left([f_i(x)]^{\lambda}[f_i(y)]^{\mu}\right) \\ &= [\prod_{i=1}^{m} f_i(x)]^{\lambda}[\prod_{i=1}^{m} f_i(y)]^{\mu} \\ &= [f(x)]^{\lambda}[f(y)]^{\mu}. \end{aligned}$$

This shows that f is a *log-p*-convex function.

It can be proved in the same way for the *log-p*-concave functions. □

Theorem 2.9. *$f : U \to \mathbb{R}_{++}$ is a log-p-convex function if and only if $\frac{1}{f}$ is a log-p-concave function.*

Proof. $(\Rightarrow)$: Let $x, y \in U$ and $\lambda, \mu \geq 0$ such that $\lambda^p + \mu^p = 1$. Then we get

$$(\frac{1}{f})(\lambda x + \mu y) = \frac{1}{f(\lambda x + \mu y)} \geq \frac{1}{[f(x)]^{\lambda}[f(y)]^{\mu}} = [(\frac{1}{f})(x)]^{\lambda}[(\frac{1}{f})(y)]^{\mu}.$$

$(\Leftarrow)$: Let $x, y \in U$ and $\lambda, \mu \geq 0$ such that $\lambda^p + \mu^p = 1$. Then we can write

$$(\frac{1}{f})(\lambda x + \mu y) = \frac{1}{f(\lambda x + \mu y)} \geq \frac{1}{[f(x)]^{\lambda}[f(y)]^{\mu}}.$$

Thus, we obtain

$$f(\lambda x + \mu y) \leq [f(x)]^{\lambda}[f(y)]^{\mu}.$$

□

By using Theorem 2.8 and Theorem 2.9 it can be obtained the following corollaries:

Corollary 2.10. *Let $f, g : U \to \mathbb{R}_{++}$. If f is a log-p-convex (log-p-concave) function and g is a log-p-concave (log-p-convex) function, then $\frac{f}{g}$ is a log-p-convex (log-p-concave) function.*

Corollary 2.11. *Let $f_i : U \to \mathbb{R}_{++}$ for $i \in \{1, 2, ..., m\}$ and $g_j : U \to \mathbb{R}_{++}$ for $j \in \{1, 2, ..., k\}$. If f_i is a log-p-convex (log-p-concave) function and g_j is a log-p-concave (log-p-convex) function, then $\frac{\prod_{i=1}^{m} f_i}{\prod_{j=1}^{k} g_j}$ is a log-p-convex (log-p-concave) function.*

Theorem 2.12. *Let $f : U \to \mathbb{R}_{++}$ and $\alpha > 0$. If f is a log-p-convex (log-p-concave) function, then f^{α} is a log-p-convex (log-p-concave) function.*

Proof. Let $\alpha > 0$, $x, y \in U$ and $\lambda, \mu \geq 0$ such that $\lambda^p + \mu^p = 1$.

Let f be a *log-p*-convex function. Then we get

$$(f^{\alpha})(\lambda x + \mu y) = [f(\lambda x + \mu y)]^{\alpha} \leq ([f(x)]^{\lambda}[f(y)]^{\mu})^{\alpha} = [(f^{\alpha})(x)]^{\lambda}[(f^{\alpha})(y)]^{\mu}.$$

The proof for *log-p*-concave functions is similar. □

Theorem 2.13. *Let $f : U \to \mathbb{R}_{++}$ and $\alpha < 0$. If f is a log-p-convex (log-p-concave) function, then f^{α} is a log-p-concave (log-p-convex) function.*

Proof. Let $\alpha < 0$, $x, y \in U$ and $\lambda, \mu \geq 0$ such that $\lambda^p + \mu^p = 1$. Let f be a *log-p*-convex function. Using $f^{\alpha} = (\frac{1}{f})^{-\alpha}$, Theorem 2.9 and Theorem 2.12.(ii) we obtain that f^{α} is a *log-p*-concave function .

The proof for *log-p*-concave functions is similar. □

By using Theorem 2.8 and Theorem 2.12 it can be obtained the following corollary.

Corollary 2.14. *Let $f_i : U \to \mathbb{R}_{++}$ for $i \in \{1, 2, ..., m\}$.*

(i) Let $\alpha > 0$. If f_i is a log-p-convex (log-p-concave) function for all $i \in \{1, 2, ..., m\}$, then $\prod_{i=1}^{m} f_i^{\alpha}$ is a log-p-convex (log-p-concave) function.

(ii) Let $\alpha < 0$. If f_i is a log-p-convex (log-p-concave) function for all $i \in \{1, 2, ..., m\}$, then $\prod_{i=1}^{m} f_i^{\alpha}$ is a log-p-concave (log-p-convex) function.

Theorem 2.15. *Let $f : U \to \mathbb{R}_{++}$.*

(i) If f is a log-p-convex function, then the function αf is log-p-convex for all $\alpha \in [0, 1]$.

(ii) If f is a log-p-concave function, then the function αf is log-p-concave for all $\alpha \geq 1$.

Proof. (i) Let $x, y \in U$, $\alpha \in [0, 1]$ and $\lambda, \mu \geq 0$ such that $\lambda^p + \mu^p = 1$. Then we get

$$\begin{aligned}
(\alpha f)(\lambda x + \mu y) &= \alpha f(\lambda x + \mu y) \\
&\leq \alpha [f(x)]^{\lambda}[f(y)]^{\mu} \\
&\leq \alpha^{\lambda + \mu}[f(x)]^{\lambda}[f(y)]^{\mu} \\
&= [(\alpha f)(x)]^{\lambda}[(\alpha f)(y)]^{\mu}.
\end{aligned}$$

(ii) The proof is similar to (i). □

Theorem 2.16. *Let $f_n : U \to \mathbb{R}_{++}$ be a log-p-convex function for all $n \in \mathbb{N}^{+}$. If the functions f_n converge pointwise to the function $f : U \to \mathbb{R}_{++}$ then f is a log-p-convex function.*

Proof. Let $x, y \in U$ and $\lambda, \mu \geq 0$ such that $\lambda^p + \mu^p = 1$. Then we have

$$\begin{aligned} f(\lambda x + \mu y) &= \lim_{n\to\infty} f_n(\lambda x + \mu y) \\ &\leq \lim_{n\to\infty} ([f_n(x)]^\lambda [f_n(y)]^\mu) \\ &= \lim_{n\to\infty} [f_n(x)]^\lambda \cdot \lim_{n\to\infty} [f_n(y)]^\mu \\ &= [\lim_{n\to\infty} f_n(x)]^\lambda \cdot [\lim_{n\to\infty} f_n(y)]^\mu \\ &= [f(x)]^\lambda [f(y)]^\mu. \end{aligned}$$

□

Theorem 2.17. *Let $0 \in U$. If $f : U \to \mathbb{R}_{++}$ is a log-p-convex function, then*
(i) $f(0) \leq 1$,
(ii) $f(\lambda x) \leq [f(x)]^\lambda$ for all $\lambda \in [0, 1]$.

Proof. Let $\lambda, \mu \geq 0$ such that $\lambda^p + \mu^p = 1$.
(i) We can write

$$f(0) = f(\lambda 0 + \mu 0) \leq [f(0)]^\lambda [f(0)]^\mu = [f(0)]^{\lambda+\mu}.$$

Taking logarithm of both sides we have $log f(0) \leq (\lambda + \mu) log f(0)$. Then we get $log f(0)[1 - (\lambda + \mu)] \leq 0$, i.e., $f(0) \leq 1$.
(ii) Using $f(0) \leq 1$, we can write

$$f(\lambda x) = f(\lambda x + \mu 0) \leq [f(x)]^\lambda [f(0)]^\mu \leq [f(x)]^\lambda.$$

This property is equivalent to be starshaped of a *log*-*p*-convex function.

□

It is known that the sum of logarithmic convex functions is also logarithmic convex [18]. Next example shows that the sum of *log*-*p*-convex functions is not necessarily *log*-*p*-convex. This fact shows a difference between *log*-*p*-convexity and *log*-convexity.

Example 2.18. *Although the function $f : \mathbb{R} \to \mathbb{R}_{++}$ defined by $f(x) = e^x$ is log-p-convex, $f + f = 2f$ is not log-p-convex. Since $(f + f)(0) = 2f(0) = 2 > 1$, using Theorem 2.17 (ii) we get that $2f$ is not log-p-convex.*

The following example shows that the composition of *log*-*p*-convex functions is not necessarily *log*-*p*-convex.

Example 2.19. *Let us consider the function $f : \mathbb{R} \to \mathbb{R}_{++}$ defined by $f(x) = e^x$. Since $(f \circ f)(0) = e > 1$, using Theorem 2.17 (i) we obtain that $f \circ f$ is not log-p-convex.*

Theorem 2.20. *Let $f : U \to \mathbb{R}_{++}$ be a p-convex (p-concave) function and $g : f(U) \to \mathbb{R}_{++}$ be a nondecreasing log-p-convex (log-p-concave) function. Then $g \circ f : U \to \mathbb{R}_{++}$ is a log-p-convex (log-p-concave).*

Proof. Let $x, y \in U$ and $\lambda, \mu \geq 0$ such that $\lambda^p + \mu^p = 1$. Then we get

$$\begin{aligned}(g \circ f)(\lambda x + \mu y) &= g(f(\lambda x + \mu y)) \\ &\leq g(\lambda f(x) + \mu f(y)) \\ &\leq [g(f(x))]^\lambda [g(f(y))]^\mu \\ &= [(g \circ f)(x)]^\lambda [(g \circ f)(y)]^\mu.\end{aligned}$$

For a *log-p*-concave function g and a p-concave function f, the *log-p*-concavity of $g \circ f$ is established in a similar way. □

Different *log-p*-convex functions can be obtained by using Theorem 2.20.

Example 2.21. *For $f(x) = x^2$ and $g(x) = e^x$ we can obtain that $(g \circ f)(x) = e^{x^2}$ is log-p-convex function.*

Theorem 2.22. *Let $f : U \to \mathbb{R}_{++}$ be a p-convex (p-concave) function and $g : f(U) \to \mathbb{R}_{++}$ be a nonincreasing log-p-concave (log-p-convex) function. Then $g \circ f : U \to \mathbb{R}_{++}$ is a log-p-concave (log-p-convex).*

Proof. Let $x, y \in U$ and $\lambda, \mu \geq 0$ such that $\lambda^p + \mu^p = 1$. Then we get

$$\begin{aligned}(g \circ f)(\lambda x + \mu y) &= g(f(\lambda x + \mu y)) \\ &\geq g(\lambda f(x) + \mu f(y)) \\ &\geq [g(f(x))]^\lambda [g(f(y))]^\mu \\ &= [(g \circ f)(x)]^\lambda [(g \circ f)(y)]^\mu.\end{aligned}$$

For a *log-p*-convex function g and a p-concave function f, the *log-p*-concavity of $g \circ f$ is established in a similar way. □

Definition 2.23. *Let $U \subseteq \mathbb{R}$. The function $f : U \to \mathbb{R}_{++}$ is called multiplicatively log-p-convex if*

$$f(x^\lambda y^\mu) \leq [f(x)]^\lambda [f(y)]^\mu$$

for all $x, y \in U$ and for all $\lambda, \mu \geq 0$ such that $\lambda^p + \mu^p = 1$.

Theorem 2.24. *Let $f : U \to \mathbb{R}_{++}$ be a log-p-convex function and $g : f(U) \to \mathbb{R}_{++}$ be a nondecreasing multiplicatively log-p-convex function. Then $g \circ f : U \to \mathbb{R}_{++}$ is a log-p-convex.*

Proof. Let $x, y \in U$ and $\lambda, \mu \geq 0$ such that $\lambda^p + \mu^p = 1$. Then we get

$$(g \circ f)(\lambda x + \mu y) = g(f(\lambda x + \mu y)) \leq g([f(x)]^\lambda [f(y)]^\mu) \leq [(g \circ f)(x)]^\lambda [(g \circ f)(y)]^\mu.$$

□

Theorem 2.25. *Let $f : U \to \mathbb{R}_{++}$, $a > 0$ and $a \neq 1$. a^f is a log-p-convex (log-p-concave) function if and only if $a > 1$ and f is a p-convex (p-concave) function, or $0 < a < 1$ and f is a p-concave (p-convex) function.*

Proof. $(\Rightarrow)$: Let $x, y \in U$ and $\lambda, \mu \geq 0$ such that $\lambda^p + \mu^p = 1$. For $a > 1$, then we can write from *log-p*-convexity of a^f

$$(a^f)(\lambda x + \mu y) = a^{f(\lambda x + \mu y)} \leq [a^{f(x)}]^{\lambda}[a^{f(y)}]^{\mu} = a^{\lambda f(x) + \mu f(y)}.$$

From $a > 1$ we obtain $f(\lambda x + \mu y) \leq \lambda f(x) + \mu f(y)$.

For $0 < a < 1$, we can obtain p-concavity of f.

$(\Leftarrow)$: This aspect of the proof can be obtained using similar considerations.

The rest of the proof can be done similarly. □

Lemma 2.26. *If $f : U \to \mathbb{R}$ is p-convex then $f - 1$ is p-convex.*

Proof. Let $x, y \in U$ and $\lambda, \mu \geq 0$ such that $\lambda^p + \mu^p = 1$. Then we get

$$\begin{aligned}(f-1)(\lambda x + \mu y) &= f(\lambda x + \mu y) - 1 \\ &\leq \lambda f(x) + \mu f(y) - (\lambda + \mu) \\ &= \lambda(f(x) - 1) + \mu(f(y) - 1) \\ &= \lambda(f-1)(x) + \mu(f-1)(y).\end{aligned}$$

□

Lemma 2.27. *If the function $f^{\frac{1}{n}} : U \to \mathbb{R}$ is p-convex function for each $n \in \mathbb{N}^+$ then the function $n(f^{\frac{1}{n}} - 1)$ is a p-convex function.*

Proof. Let $n \in \mathbb{N}^+$, $x, y \in U$ and $\lambda, \mu \geq 0$ such that $\lambda^p + \mu^p = 1$. Then we obtain

$$\begin{aligned}(n(f^{\frac{1}{n}} - 1))(\lambda x + \mu y) &= n(f^{\frac{1}{n}} - 1)(\lambda x + \mu y) \\ &\leq n(\lambda(f^{\frac{1}{n}} - 1)(x) + \mu(f^{\frac{1}{n}} - 1)(y)) \\ &= \lambda(n(f^{\frac{1}{n}} - 1))(x) + \mu(n(f^{\frac{1}{n}} - 1))(y).\end{aligned}$$

□

Lemma 2.28. *Let $f_n : U \to \mathbb{R}$ be p-convex for all $n \in \mathbb{N}^+$. If the functions f_n converge pointwise to the function f then f is p-convex.*

Proof. Let $x, y \in U$ and $\lambda, \mu \geq 0$ such that $\lambda^p + \mu^p = 1$. Then we have

$$\begin{aligned}f(\lambda x + \mu y) &= \lim_{n\to\infty} f_n(\lambda x + \mu y) \\ &\leq \lim_{n\to\infty} (\lambda f_n(x) + \mu f_n(y)) \\ &= \lambda \lim_{n\to\infty} f_n(x) + \mu \lim_{n\to\infty} f_n(y) \\ &= \lambda f(x) + \mu f(y).\end{aligned}$$

□

Using the above three lemmas, the following important theorem is obtained.

Theorem 2.29. *Let $f : U \to \mathbb{R}_{++}$. If $f^{\frac{1}{n}}$ is p-convex for all $n \in \mathbb{N}^+$, then f is log-p-convex.*

Proof. Let $f^{\frac{1}{n}}$ be p-convex for all $n \in \mathbb{N}^+$. From Lemma 2.26, $f^{\frac{1}{n}} - 1$ is p-convex for all $n \in \mathbb{N}^+$. Using Lemma 2.27, we have that $g_n = n(f^{\frac{1}{n}} - 1)$ is p-convex for all $n \in \mathbb{N}^+$. From Lemma 2.28, $\lim_{n\to\infty} g_n = logf$ is p-convex. Hence f is log-p-convex.

□

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